

# THE ARITHMETICITY OF A KODAIRA FIBRATION IS DETERMINED BY ITS UNIVERSAL COVER

GABINO GONZÁLEZ-DIEZ AND SEBASTIÁN REYES-CARROCCA

**ABSTRACT.** Let  $S \rightarrow C$  be a Kodaira fibration. Here we show that whether or not the algebraic surface  $S$  is defined over a number field depends only on the biholomorphic class of its universal cover.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

Let  $X \subset \mathbb{P}^n$  be a complex projective variety and  $k$  a subfield of the field of the complex numbers  $\mathbb{C}$ . We shall say that  $X$  is defined over  $k$  or that  $k$  is a *field of definition* for  $X$  if there exists a collection of homogenous polynomials  $f_0, \dots, f_m$  with coefficients in  $k$  so that the variety they define is isomorphic to  $X$ . We will say that  $X$  is *arithmetic* if it is defined over  $\overline{\mathbb{Q}}$  or equivalently over a number field.

While it is classically known that there are only three simply connected Riemann surfaces, there is a huge amount of possibilities for the holomorphic universal cover of a complex surface  $S$ . It would be interesting to understand the extent to which the arithmeticity of a projective surface can be read off from its holomorphic universal cover. In this short note we study this question for a very important class of complex surfaces known in the literature as Kodaira fibrations.

A *Kodaira fibration* consists of a non-singular compact complex surface  $S$ , a compact Riemann surface  $C$  and a surjective holomorphic map  $S \rightarrow C$  everywhere of maximal rank such that the fibers are connected and not mutually isomorphic Riemann surfaces. The genera  $g$  of the fibre and  $b$  of  $C$  are called the genus of the fibration and of the base respectively. It is known that such a surface  $S$  must be an algebraic surface of general type and that necessarily  $g \geq 3$  and  $b \geq 2$ . We notice that an important theorem by Arakelov [1] implies that, up to isomorphism, there are only finitely many Kodaira fibrations over a given algebraic curve  $C$ .

In 1967, Kodaira [13] used fibrations of this kind to show that the signature of a differentiable fiber bundle need not be multiplicative. Soon after Kas [12] studied the deformation space of the surfaces constructed by Kodaira, and two years later Atiyah [2] and Hirzebruch [10] studied further

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properties concerning the signature of Kodaira fibrations in a volume dedicated to Kodaira himself.

Explicit constructions of Kodaira surfaces have been made by González-Diez and Harvey [8], Bryan and Donagi [4], Zaal [16] and Catanese and Rollenske [6].

We now state the main results of the paper

**Theorem 1.** *Let  $k$  be an algebraically closed subfield of the complex numbers and  $S_1 \rightarrow C_1$  and  $S_2 \rightarrow C_2$  two Kodaira fibrations so that their respective holomorphic universal covers are biholomorphically equivalent. Then  $S_1$  is defined over  $k$  if and only if  $S_2$  is defined over  $k$ . In particular,  $S_1$  is arithmetic if and only if  $S_2$  is arithmetic.*

To prove this theorem we will have to show first the following result which is interesting in its own right

**Theorem 2.** *Let  $k$  be an algebraically closed subfield of the complex numbers and  $S \rightarrow C$  a Kodaira fibration. Then  $S$  is defined over  $k$  if and only if  $C$  is defined over  $k$ . In particular,  $S$  is an arithmetic surface if and only if  $C$  is an arithmetic curve.*

Theorem 1 states that the arithmeticity of a Kodaira fibration can be recognized in its holomorphic universal cover. We anticipate that the holomorphic universal cover of  $S$  is a contractible bounded domain  $\mathcal{B} \subset \mathbb{C}^2$  (see Section 2). Clearly, Theorem 1 implies that the biholomorphism class of  $\mathcal{B}$  varies together with the variation of  $S$  in moduli space. We note that in general Kodaira surfaces are not rigid ([12], [6]).

## 2. UNIFORMIZATION OF KODAIRA SURFACES

It is well-known that the universal cover of a Riemann surface is biholomorphically equivalent to the projective line  $\mathbb{P}^1$ , the complex plane  $\mathbb{C}$  or the upper half-plane  $\mathbb{H}$ . Understanding universal covers of complex manifolds of higher dimension seems to be a very complicated task. However, thanks to the work of Bers [3] and Griffiths [9] on uniformization of algebraic varieties, it is possible to describe the universal cover of a Kodaira fibration  $f : S \rightarrow C$  in a very explicit way.

Let  $\pi : \mathbb{H} \rightarrow C$  be the universal covering map of  $C$  and  $\Gamma$  the covering group so that  $C \cong \mathbb{H}/\Gamma$ . By considering the pull-back  $h : \pi^*S \rightarrow \mathbb{H}$  of  $f$  by  $\pi$ , we obtain a new fibration in which, for each  $t \in \mathbb{H}$ , the fiber  $h^{-1}(t)$  agrees with the Riemann surface  $f^{-1}(\pi(t))$ . Teichmüller theory enables us to choose uniformizations  $h^{-1}(t) = D_t/K_t$  possessing the following properties:

- (a)  $K_t$  is a Kleinian group acting on a bounded domain  $D_t$  of  $\mathbb{C}$  which is biholomorphically equivalent to a disk.
- (b) The union of all these disks  $\mathcal{B} := \cup_{t \in \mathbb{H}} D_t$  is a contractible bounded domain of  $\mathbb{C}^2$  which is biholomorphic to the universal cover of  $S$ , that is,  $S \cong \mathcal{B}/\mathbb{G}$ , where  $\mathbb{G} < \text{Aut}(\mathcal{B})$  is the covering group.

- (c) The group  $\mathbb{G}$  is endowed with a surjective homomorphism of groups  $\Theta : \mathbb{G} \rightarrow \Gamma$  which induces an exact sequence of groups

$$1 \longrightarrow \mathbb{K} \longrightarrow \mathbb{G} \xrightarrow{\Theta} \Gamma \longrightarrow 1$$

where, for each  $t \in \mathbb{H}$ , the subgroup  $\mathbb{K}$  preserves  $D_t$  and acts on it as the Kleinian group  $K_t$ .

We note that  $\mathcal{B}$  carries itself a fibration structure  $\mathcal{B} \rightarrow \mathbb{H}$  whose fiber over  $t \in \mathbb{H}$  is  $D_t$  (i.e.  $\mathcal{B}$  is a *Bergman domain* in Bers' terminology).

In [14] and [15] Shabat studied the automorphism groups of universal covers of families of Riemann surfaces and proved a deep result which in the case of Kodaira fibrations amounts to the following theorem.

**Theorem (Shabat)** Let  $f : S \rightarrow C$  be a Kodaira fibration and  $\mathcal{B}$  the holomorphic universal cover of  $S$ . Then:

- (a) the covering group  $\mathbb{G}$  of  $S$  has finite index in  $\text{Aut}(\mathcal{B})$ .
- (b)  $\text{Aut}(\mathcal{B})$  is a discrete group.

### 3. PROOF OF THEOREMS 1 AND 2

We denote by  $\text{Gal}(\mathbb{C})$  the group of field automorphisms of  $\mathbb{C}$ . The natural action of  $\text{Gal}(\mathbb{C})$  on the ring of polynomials  $\mathbb{C}[x_0, \dots, x_n]$  induces a well-defined action  $(\sigma, X) \rightarrow X^\sigma$  on the set of isomorphism classes of algebraic varieties. Throughout this section  $k$  will denote an algebraically closed subfield of  $\mathbb{C}$  and  $\text{Gal}(\mathbb{C}/k)$  the subgroup of  $\text{Gal}(\mathbb{C})$  consisting of all automorphisms  $\sigma$  fixing the elements of  $k$ .

**3.1. Proof of Theorem 2.** Let  $f : S \rightarrow C$  be a Kodaira fibration. Let us assume that the curve  $C$  is defined over  $k$ . Then  $C^\sigma = C$  for all  $\sigma \in \text{Gal}(\mathbb{C}/k)$ , and so, by Arakelov's finiteness Theorem, there are only finitely many pairwise non-isomorphic Kodaira fibrations  $f^\sigma : S^\sigma \rightarrow C^\sigma$  with  $\sigma \in \text{Gal}(\mathbb{C}/k)$ . This implies that  $S$  is defined over  $k$  [7, Crit. 2.1].

In order to prove the converse, we begin by recalling that a complex manifold  $X$  is named *hyperbolic* if every holomorphic map  $\mathbb{C} \rightarrow X$  is a constant map. We claim that Kodaira fibrations are hyperbolic. In fact, let  $f : S \rightarrow C$  be a Kodaira fibration and  $\varphi : \mathbb{C} \rightarrow S$  a non-constant holomorphic map. As  $C$  has genus greater than one, the map  $f \circ \varphi : \mathbb{C} \rightarrow C$  must be constant and therefore the image of  $\varphi$  has to be contained in a fiber  $f^{-1}(x)$  for some  $x \in C$ . Since the fibers are also hyperbolic,  $\varphi$  must be constant too.

Let us now assume that  $S$  is defined over  $k$ , so that  $S^\sigma = S$  for any  $\sigma \in \text{Gal}(\mathbb{C}/k)$ . Now as  $S$  is a Kähler hyperbolic manifold, the canonical divisor  $K_S$  is ample [5] and this implies that only finitely many curves  $R$  of genus greater than one can arise as the image of a surjective morphism  $S \rightarrow R$  [11]. In particular the family  $\{C^\sigma : \sigma \in \text{Gal}(\mathbb{C}/k)\}$  itself contains only finitely many isomorphism classes of curves. It then follows that  $C$  is defined over  $k$  [7, Crit. 2.1], as required.

**3.2. Proof of Theorem 1.** Let  $f_2 : S_2 \rightarrow C_2$  be a Kodaira fibration and  $S_1$  an arbitrary non-singular complex surface. Let us denote by  $\mathcal{B}_i$  the universal cover of  $S_i$  and suppose that there exists an isomorphism  $\alpha : \mathcal{B}_1 \rightarrow \mathcal{B}_2$  between them. Let  $\mathbb{G}_i$  be the uniformizing group of  $S_i$  so that  $\mathcal{B}_i/\mathbb{G}_i \cong S_i$ . By Shabat's Theorem  $\mathbb{G}_2$  has finite index in  $\text{Aut}(\mathcal{B}_2)$ . We claim that  $\mathbb{G}_1$  has finite index in  $\text{Aut}(\mathcal{B}_1)$  too. In fact, as  $\mathcal{B}_1/\text{Aut}(\mathcal{B}_1) \cong \mathcal{B}_2/\text{Aut}(\mathcal{B}_2)$  and as  $\text{Aut}(\mathcal{B}_2)$  is a discrete group, the projection map  $S_1 = \mathcal{B}_1/\mathbb{G}_1 \rightarrow \mathcal{B}_1/\text{Aut}(\mathcal{B}_1)$  is a holomorphic map between (normal) compact complex surfaces; from here the claim follows.

By replacing  $\mathbb{G}_1$  by  $\alpha\mathbb{G}_1\alpha^{-1}$  we can assume that  $\mathcal{B}_1 = \mathcal{B}_2$ , so we denote  $\mathcal{B}_i$  simply by  $\mathcal{B}$ . As both  $\mathbb{G}_1$  and  $\mathbb{G}_2$  have finite index in  $\text{Aut}(\mathcal{B})$ , so must do their intersection  $\mathbb{G}_{12} = \mathbb{G}_1 \cap \mathbb{G}_2$ . The complex surface  $S_{12} := \mathcal{B}/\mathbb{G}_{12}$  is endowed with two finite degree covers  $\pi'_i : S_{12} \rightarrow S_i$  with  $i = 1, 2$ ; in particular,  $S_{12}$  is also a projective surface. Moreover, if we denote by  $\Theta_{12}$  the restriction of the epimorphism  $\Theta : \mathbb{G}_2 \rightarrow \Gamma_2$  introduced in the previous section to  $\mathbb{G}_{12}$ , then we obtain an exact sequence of groups

$$1 \longrightarrow \mathbb{K}_{12} \longrightarrow \mathbb{G}_{12} \xrightarrow{\Theta_{12}} \Gamma_{12} \longrightarrow 1$$

where  $\Gamma_{12} = \Theta_{12}(\mathbb{G}_{12})$  and  $\mathbb{K}_{12} = \ker(\Theta_{12}) = \mathbb{K} \cap \mathbb{G}_{12}$ . As in Section 2, this sequence defines a Kodaira fibration  $f_{12} : S_{12} \rightarrow C_{12} := \mathbb{H}/\Gamma_{12}$  whose fiber over  $[t]_{\Gamma_{12}}$  is isomorphic to the Riemann surface  $D_t/K_t^{12}$  where  $K_t^{12}$  is the Kleinian group that realizes the action of  $\mathbb{K}_{12}$  on  $D_t$ . We have the following commutative diagram

$$\begin{array}{ccc} & \mathcal{B} & \\ & \swarrow \quad \searrow & \\ S_{12} & \xrightarrow{\pi'_2} & S_2 \\ f_{12} \downarrow & & \downarrow f \\ C_{12} & \xrightarrow{p} & C_2 \end{array}$$

where  $p$  is the projection induced by the finite index inclusion  $\Gamma_{12} < \Gamma_2$ .

Let us now assume that  $S_2$  is defined over  $k$ . Then Theorem 2 ensures that  $C_2$  is also defined over  $k$ . Furthermore, being an unbranched cover of  $C_2$ , the curve  $C_{12}$  must also be defined over  $k$  [7, Th. 4.1]. Again, by Theorem 2 we conclude that  $S_{12}$  is defined over  $k$ . Now, as  $S_1$  is a surface of general type arising as the image (by  $\pi'_1$ ) of a surface defined over  $k$ , it must be defined over  $k$  as well [7, Prop. 3.2]. This proves Theorem 1.

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DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA DE MADRID.  
*E-mail address:* gabino.gonzalez@uam.es, sereyesc@gmail.com